

An ADM 3+1 formulation for Smooth Lattice General Relativity

Leo Brewin

Department of Mathematics

Monash University

Clayton, Vic. 3168

Australia

Abstract

A new hybrid scheme for numerical relativity will be presented. The scheme will employ a 3-dimensional spacelike lattice to record the 3-metric while using the standard 3+1 ADM equations to evolve the lattice. Each time step will involve three basic steps. First, the coordinate quantities such as the Riemann and extrinsic curvatures are extracted from the lattice. Second, the 3+1 ADM equations are used to evolve the coordinate data, and finally, the coordinate data is used to update the scalar data on the lattice (such as the leg lengths). The scheme will be presented only for the case of vacuum spacetime though there is no reason why it could not be extended to non-vacuum spacetimes. The scheme allows any choice for the lapse function and shift vectors. An example for the Kasner T^3 cosmology will be presented and it will be shown that the method has, for this simple example, zero discretisation error.

1. Introduction

In a recent paper [1] a new numerical scheme was developed and successfully applied in the construction of time symmetric initial data for a Schwarzschild black hole. The method was based on two key ideas, one to use short geodesic segments as the legs of the lattice and two, to employ Riemann normal coordinates local to each vertex.

Our primary aim in this paper is to present a hybrid scheme in which a 3-dimensional spacelike lattice can be evolved using the standard ADM vacuum 3+1 equations.

The general scheme envisaged in [1] can be summarised as follows. The lattice is a network of vertices and legs and is fully specified by the connectivity matrix and the leg lengths. To each vertex there is an associated computational cell defined by some local sub-set of the lattice (eg. the legs and vertices of the tetrahedra attached to this vertex, though larger subsets could be considered). Riemann normal coordinates are employed in each computational cell leading to a metric of the form

$$g_{\mu\nu}(x) = g_{\mu\nu} - \frac{1}{3}R_{\mu\alpha\nu\beta}x^\alpha x^\beta + \mathcal{O}(\epsilon^3) \quad (1.1)$$

where $g_{\mu\nu}$ is chosen to be $\text{diag}(1, 1, 1)$ and ϵ is a typical length scale for the computational cell. The crucial assertion that the legs are geodesic segments of this metric leads directly

to the following equation for the leg length L_{ij} between vertices (i) and (j)

$$L_{ij}^2 = g_{\mu\nu} \Delta x_{ij}^\mu \Delta x_{ij}^\nu - \frac{1}{3} R_{\mu\alpha\nu\beta} x_i^\mu x_i^\nu x_j^\alpha x_j^\beta + \mathcal{O}(\epsilon^5) \quad (1.2)$$

where $\Delta x_{ij}^\mu = x_j^\mu - x_i^\mu$. For each computational cell there would be a small set of such equations, which in [1] were referred to as the smooth lattice equations. These could, in principle, be solved as a coupled set of equations for (estimates of) the Riemann tensor and any coordinates not fixed by gauge freedoms. The computations for any one cell are fully decoupled from those of any other cell. Thus this is a local problem and should be amenable to Newton-Raphson methods. It is at this point that the design of the lattice becomes crucial. It must be such that the equations in each cell yield unambiguous estimates for the Riemann tensor. If the system is singular then there will exist continuous families of solutions for the curvatures. This indicates that the lattice lacks sufficient information to tie down a unique estimate (or at worst a finite discrete set of estimates) for the curvatures. The solution then would be to add extra information to the lattice such as extra leg lengths. The lattice chosen in [1] did not suffer from this problem.

However, subsequent experiments [2] have shown that many seemingly suitable lattices (such as in Fig 4) are singular. Following these experiments a slight variation to the above scheme has proved to be very successful. It entails the addition of extra information in the form of the polar directions of the legs attached to the central vertex of the cell. Since the connection vanishes at the origin (by definition of the RNC) this is equivalent to specifying the spherical polar angles of the vertices at the ends of these legs. This not only supplies the missing information but it also fully decouples the set of smooth lattice equations.

The modified scheme works as follows. Each computational cell will, for the remainder of this paper, be assumed to consist of the central vertex and the tetrahedra attached to this vertex. For each cell there will be one central vertex, for which we will always use the label 0, and a set of vertices on the surface of the cell. To each surface vertex (i) we assign the polar coordinates (θ_i, ϕ_i) . With this, the equations (1.2) for the radial legs are satisfied by

$$\begin{aligned} x_i &= L_{oi} \cos \phi_i \sin \theta_i \\ y_i &= L_{oi} \sin \phi_i \sin \theta_i \\ z_i &= L_{oi} \cos \theta_i \end{aligned} \quad (1.3)$$

where L_{oi} is the length of the radial leg from the origin of the cell to the surface vertex (i) . The (x_i, y_i, z_i) are the Riemann normal coordinates for vertex (i) . Next, the curvatures can be computed by solving, by a least squares method, the leg length equations (1.2) applied to the legs on the surface of the computational cell. Least squares, or some other method, will be necessary since these equations are almost certainly overdetermined.

In this scheme the basic data given on the lattice are the L_{ij}, θ_i and ϕ_i from which we compute, by the above equations, the x_i^μ and the $R_{\mu\nu\alpha\beta}$.

Note that for each leg on the surface of the cell there is an associated angle subtended at the origin of the cell. In terms of the spherical polar coordinates for the vertices (i) and (j) of the leg (ij) , this angle, α_{ij} , can be computed from

$$\cos \alpha_{ij} = \cos \theta_i \cos \theta_j + \cos(\phi_i - \phi_j) \sin \theta_i \sin \theta_j \quad (1.4)$$

and in terms of the Riemann normal coordinates from

$$\cos \alpha_{ij} = g_{\mu\nu} \frac{\Delta x_{oi}^\mu \Delta x_{oj}^\nu}{L_{oi} L_{oj}} \quad (1.5)$$

Both of these equations will be used later in this paper.

Note that in the construction of our Riemann normal coordinates we have taken advantage of some coordinate freedoms. For example we have used translational freedoms to tie the origin of the RNC to the central vertex. We have also chosen our coordinate axes to be orthogonal at the origin (hence $g_{\mu\nu} = \text{diag}(1, 1, 1)$). However, there are some residual gauge freedoms, namely the three degrees of freedom to rotate the coordinate axes about the origin. This freedom can be accounted for by specifying three of the polar angles θ_i and ϕ_i , such as θ_1, ϕ_1 and θ_2 .

2. Smooth lattice kinematics

In the familiar 3+1 formulation of general relativity a typical spacetime can be represented in two equivalent ways, one as an evolving 3-metric on 3-dimensional manifold and two, as sequence of distinct 3-dimensional hypersurfaces of the spacetime. In the later picture, each hypersurface records one instant in the evolution of the 3-metric. The information that one must specify to fully define the 4-metric on the spacetime are the 3-metric, the lapse function and the shift vectors on each hypersurface. In this section we shall adapt this picture to one in which each hypersurface is a smooth 3-dimensional lattice.

The general idea will be to allow the basic data for each lattice, namely the L_{ij}, θ_i and ϕ_i , to be functions of time. On each hypersurface we will continue to employ Riemann normal coordinates and the above equations for computing the curvature components. Since the lattice data are now presumed to be functions of time so too must the coordinates x_i^μ and the curvature components $R_{\mu\nu\alpha\beta}$. We will also impose our previously stated gauge conditions on each hypersurface, in particular that $g_{\mu\nu} = \text{diag}(1, 1, 1)$ and that the origin of the RNC remains forever tied to the central vertex of the cell. One could choose a dynamical set of gauge conditions but to do so at this stage would be an un-necessary complication.

It is customary to use Latin indices to denote spatial components (eg. g_{ij} for the 3-metric) and Greek indices for spacetime quantities (eg. $g_{\mu\nu}$ for the 4-metric). As we would like to use Latin indices to denote simplices of the lattice we shall choose to use Greek indices for spatial components. Thus in the following $g_{\mu\nu}, R_{\mu\nu\alpha\beta}$ etc. will denote 3-tensors that live in each 3-dimensional hypersurface.

We will be making frequent reference to the current and future hypersurfaces and the various structures that define and link these hypersurfaces. We will use Σ_0 to represent the current hypersurface and $\Sigma_{\delta t}$ to represent the hypersurface just slightly to the future of Σ_0 . The time coordinate t is constant on each hypersurface with $t = t_0$ on Σ_0 and $t = t_0 + \delta t$ on $\Sigma_{\delta t}$. The lapse function will be denoted by N and the shift vector by N^μ . Both the lapse function and shift vectors will be defined on the vertices of the lattice. Note that even though we are using Riemann normal coordinates on each hypersurface the 4-dimensional coordinates (t, x^μ) will not, in general, be in Riemann normal form.

Consider any 3-dimensional smooth lattice and focus attention on one vertex. Through this vertex one can construct three well defined curves – the timelike worldline of the vertex, the timelike geodesic tangent to the normal vector at the vertex and the timelike worldline of the observer with constant spatial coordinates. It is important to realise that these may be three distinct curves. Thus upon extending these curves from their common origin in Σ_0 to $\Sigma_{\delta t}$ we can expect to obtain three new points in $\Sigma_{\delta t}$. This situation is depicted in Fig (1). In the usual 3+1 picture only two of these curves, the normal geodesic and the observers worldline, enter into the discussion. In the smooth lattice we must also consider the freedom for vertices to drift relative to these two curves. For this reason we will introduce a new vector, the drift vector γ^μ .

From Fig (1) we can clearly see that the shift and drift vectors at vertex (i) are related by

$$\frac{dx_i^\mu}{dt} = -\gamma_i^\mu + N_i^\mu \quad (2.1)$$

where $x_i^\mu(t)$ are the time dependent coordinates of vertex (i) . The dx_i^μ/dt can be easily computed from (1.3). Note that in our chosen gauge $x_i^\mu = dx_i^\mu/dt = 0$ at the origin of the RNC.

One now has a choice as to which of the drift or shift vectors should be freely specified on the lattice. Since the drift vector has a strong geometric appeal we shall choose to freely specify it and to use the above equation (2.1) to compute the shift vector. A convenient choice is to set the drift vector to be zero at the origin (ie. the vertex at the origin of the RNC in Σ_0 is evolved along the timelike normal). Though, of course, there may be occasions where such a choice is not appropriate.

We will now introduce a second lattice which will help us later on in making the transition to the standard ADM 3+1 equations. Consider one hypersurface and its associated lattice. Consider now a second lattice coincident with the original lattice. We will refer to these as the primary and shadow lattices respectively. The shadow lattice will be chosen so such that its vertices are evolved along the normals to the hypersurface (ie. as if the drift vectors were set to zero everywhere). We do this because the ADM 3+1 equations take on a particularly simple form when expressed as time derivatives along the normals. The shadow lattice will only be employed for one time step. Upon completion of the time step the current shadow lattice will be discarded and a new shadow lattice created coinciding with the updated primary lattice. Each of these shadow lattices are introduced only as an aid in the exposition of our algorithm – they need never be created in any computer program.

Our current task is to see how we should specify the data on the shadow lattice so that it evolves along the normals.

On the primary lattice we have various quantities such as L_{ij}, θ_i, ϕ_i and x_i^μ all of which we assume to be functions of time t . Their counterparts on the shadow lattice will be denoted by the addition of a dash (ie. $L'_{ij}, x_i'^\mu$ etc.). The dash on the $x_i'^\mu$ should not be confused with coordinate transformation – the $x_i'^\mu$ are just the x^μ coordinates of vertex (i'). On the initial hypersurface Σ_0 all of the corresponding dashed and un-dashed quantities are equal (ie. $L'_{ij} = L_{ij}, x_i'^\mu = x_i^\mu$) though they may drift apart between successive hypersurfaces.

By inspection of Fig (1) one can easily verify that

$$\frac{dx_i'^\mu}{dt} = \frac{dx_i^\mu}{dt} - \gamma_i^\mu \quad (2.2)$$

Consider now equation (1.2) applied to both lattices,

$$\begin{aligned} L_{ij}^2(t) &= g_{\mu\nu} \Delta x_{ij}^\mu \Delta x_{ij}^\nu - \frac{1}{3} R_{\mu\alpha\nu\beta} x_i^\mu x_i^\nu x_j^\alpha x_j^\beta \\ L'_{ij}{}^2(t) &= g_{\mu\nu} \Delta x_{ij}'^\mu \Delta x_{ij}'^\nu - \frac{1}{3} R_{\mu\alpha\nu\beta} x_i'^\mu x_i'^\nu x_j'^\alpha x_j'^\beta \end{aligned}$$

All of the terms on the right hand side should be considered to be functions of time (except the $g_{\mu\nu}$ which we choose to be $\text{diag}(1, 1, 1)$). Differentiating with respect to t and using the above equation for $dx_i'^\mu/dt$ we obtain on Σ_0

$$\frac{dL'_{ij}}{dt} = \frac{dL_{ij}}{dt} - \frac{1}{L_{ij}} g_{\mu\nu} \Delta \gamma_{ij}^\mu \Delta x_{ij}^\nu + \frac{1}{3L_{ij}} R_{\mu\alpha\nu\beta} (\gamma_i^\mu x_i^\nu x_j^\alpha x_j^\beta + x_i^\mu x_i^\nu \gamma_j^\alpha x_j^\beta) \quad (2.3)$$

where $\Delta \gamma_{ij}^\mu = \gamma_j^\mu - \gamma_i^\mu$.

A similar argument can be applied to the angles α_{ij} . Starting with

$$\cos \alpha_{ij}(t) = g_{\mu\nu} \frac{\Delta x_{oi}^\mu \Delta x_{oj}^\nu}{L_{oi} L_{oj}} \quad (2.4)$$

$$\cos \alpha'_{ij}(t) = g_{\mu\nu} \frac{\Delta x_{oi}'^\mu \Delta x_{oj}'^\nu}{L'_{oi} L'_{oj}} \quad (2.5)$$

we obtain, by direct differentiation, that on Σ_0

$$\begin{aligned} \frac{d\alpha'_{ij}}{dt} &= \frac{d\alpha_{ij}}{dt} + \frac{\cos \alpha_{ij}}{\sin \alpha_{ij}} \left(\frac{1}{L_{oi}} \frac{d}{dt} (L'_{oi} - L_{oi}) + \frac{1}{L_{oj}} \frac{d}{dt} (L'_{oj} - L_{oj}) \right) \\ &\quad + \frac{g_{\mu\nu}}{L_{oi} L_{oj} \sin \alpha_{ij}} \left(\Delta \gamma_{oi}^\mu \Delta x_{oj}^\nu + \Delta \gamma_{oj}^\mu \Delta x_{oi}^\nu \right) \end{aligned}$$

This can be simplified by using the above equation for dL'/dt and noting that in our gauge $x_o^\mu = 0$ on Σ_0 . The result is that on Σ_0

$$\begin{aligned} \frac{d\alpha'_{ij}}{dt} &= \frac{d\alpha_{ij}}{dt} + \frac{g_{\mu\nu}}{L_{oi} L_{oj} \sin \alpha_{ij}} \left(\left(\Delta \gamma_{oi}^\mu \Delta x_{oj}^\nu + \Delta \gamma_{oj}^\mu \Delta x_{oi}^\nu \right) \right. \\ &\quad \left. - \cos \alpha_{ij} \left(\frac{L_{oj}}{L_{oi}} \Delta \gamma_{oi}^\mu \Delta x_{oi}^\nu + \frac{L_{oi}}{L_{oj}} \Delta \gamma_{oj}^\mu \Delta x_{oj}^\nu \right) \right) \end{aligned} \quad (2.6)$$

The dL'_{ij}/dt and the $d\alpha'_{ij}/dt$ measure rates of change along the normals to Σ_0 . They should therefore be expressible in terms of the extrinsic curvatures $K'_{\mu\nu}$.

Consider the shadow lattice and the coordinates x^μ . Let us introduce a second coordinate system x''^μ such that the x''^μ are constant along each worldline of the vertices of the shadow lattice. This clearly establishes a time dependent coordinate transformation between x^μ and x''^μ . Note that on Σ_0 both coordinate systems are identical. Note also that on subsequent hypersurfaces the x''^μ need not be in Riemann normal coordinate form.

Since the worldlines of the shadow lattice are tangent to the normals to Σ_0 we must have

$$\frac{dg''_{\mu\nu}}{dt} = -2N K''_{\mu\nu}$$

In the x''^μ coordinates the $L'_{ij}(t)$ and $\alpha'_{ij}(t)$ may be estimated from

$$L'^2_{ij}(t) = g''_{\mu\nu}(t) \Delta x_{ij}''^\mu \Delta x_{ij}''^\nu + \mathcal{O}(\epsilon^4) \quad (2.7)$$

$$L'_{oi} L'_{oj} \cos \alpha'_{ij}(t) = g''_{\mu\nu}(t) \Delta x_{oi}''^\mu \Delta x_{oj}''^\nu + \mathcal{O}(\epsilon^4) \quad (2.8)$$

where $g''_{\mu\nu}$ is evaluated on the geodesic tangent to the normal to Σ_0 at the origin of the RNC. Since the x''^μ need not be in Riemann normal coordinate form we can expect these estimates

to be less accurate than their Riemann normal coordinate counterparts (eg. equations (1.2) and (1.5)). This is the price we pay for not using Riemann normal coordinates. Increased accuracy could be obtained for the $L'_{ij}(t)$ by evaluating the right hand side at the centre of the legs. This then forces us to explicitly account for variations of $K_{\mu\nu}$ throughout the cell. It is not clear how one might make a similar ‘centred’ approximation for the angles $\alpha'_{ij}(t)$. Rather than trying to resolve these issues in this paper we shall settle on the simple estimates given above.

Differentiating the previous two equations with respect to time and using the above equation for $K''_{\mu\nu}$ we obtain, on Σ_0 ,

$$L'_{ij} \frac{dL'_{ij}}{dt} = -(NK_{\mu\nu})_o \Delta x^\mu_{ij} \Delta x^\nu_{ij} \quad (2.9)$$

$$\frac{d}{dt} (L'_{oi} L'_{oj} \cos \alpha'_{ij}) = -2(NK_{\mu\nu})_o \Delta x^\mu_{oi} \Delta x^\nu_{oj} \quad (2.10)$$

This system of equations for the six $K_{\mu\nu}$ at the origin of the RNC will, for almost all lattices, be highly overdetermined (there being many more L_{ij} and α_{ij} than the six $K_{\mu\nu}$). This problem can be overcome in one of two ways. We could reject all but six of the above equations. Or we could use all of the equations in a least squares estimation of the six $K_{\mu\nu}$. We will assume in the following that the least squares method has been used. In fact we will find that this issue of overdeterminism will crop up on a number of occasions in this paper. In all cases we will resort to a least squares solution as it is systematic and easy to implement in a computer program.

At this point we are in a position to calculate both the extrinsic and intrinsic curvatures tensors, at the origin of the RNC, given just the basic lattice data, namely the L_{ij}, θ_i, ϕ_i , their time derivatives and for any choice of chosen lapse function and shift and drift vectors. However we have not discussed how $K_{\mu\nu|\alpha}$ and $N_{|\mu\nu}$ may be calculated (both of which will be needed in the next section).

Consider first the calculation of $N_{|\mu\nu}$. Since N is a scalar function we have $N_{|\mu} = \partial N / \partial x^\mu$ which we can approximate at the centre of the leg (ij) by

$$(N_{|\mu})_{ij} = \frac{N_i - N_j}{L_{ij}} \Delta x^\mu_{ij} \quad (2.11)$$

Such an expression can be formed at the centre of each leg of the computational cell. From this data we can form a linear approximation $\tilde{N}_{|\mu}(x^\alpha)$ to $N_{|\mu}(x^\alpha)$

$$\tilde{N}_{|\mu}(x^\alpha) = \tilde{N}_{|\mu} + \tilde{N}_{|\mu\nu} x^\nu \quad (2.12)$$

where the constants $\tilde{N}_{|\mu}$ and $\tilde{N}_{|\mu\nu} = \tilde{N}_{|\nu\mu}$ are obtained by a least squares fit. We take the $\tilde{N}_{|\mu\nu}$ as our estimates of $N_{|\mu\nu}$ at the origin of the RNC. Similarly the $\tilde{N}_{|\mu}$ could be used as

an approximation to $N_{|\mu}$ at the origin of the RNC (though we will not need to do so). The appropriate least squares sum is chosen to be

$$S(\tilde{N}_{|\mu}, \tilde{N}_{|\mu\nu}) = \sum_{\mu} \sum_{ij} \left((N_{|\mu})_{ij} - \tilde{N}_{|\mu} - \frac{1}{2} \tilde{N}_{|\mu\nu} (x_i^{\nu} + x_j^{\nu}) \right)^2$$

Note that the symmetry condition $\tilde{N}_{|\mu\nu} = \tilde{N}_{|\nu\mu}$ must be explicitly imposed in this sum.

Do we have enough data to compute the nine numbers $\tilde{N}_{|\mu}$ and $\tilde{N}_{|\mu\nu}$? The answer is yes – each leg gives us three samples for $N_{|\mu}$ and each computational cell is certain to have more than three legs.

The computation of the $K_{\mu\nu|\alpha}$ can be performed in a similar manner. The first thing to do is to compute the $K_{\mu\nu}$ at the origin of each RNC for each computational cell. Now consider one specific computational cell. The vertices on the surface of the cell will themselves be the origins of neighbouring cells. Each of these cells carries their own estimates of the $K_{\mu\nu}$ in their own coordinates. As there is a well defined coordinate transformation between these neighbouring frames it is possible to obtain estimates of the $K_{\mu\nu}$ on each of the vertices in our chosen cell and in the coordinates of that chosen cell (as indicated in Fig (2)). We can then form a linear approximation to each of the six $K_{\mu\nu}$ in the chosen cell. Thus we put

$$\tilde{K}_{\mu\nu}(x^{\alpha}) = \tilde{K}_{\mu\nu} + \tilde{K}_{\mu\nu|\alpha} x^{\alpha}$$

with the constants $\tilde{K}_{\mu\nu}$ and $\tilde{K}_{\mu\nu|\alpha}$ obtained by another application of the least squares method. Finally we take the $\tilde{K}_{\mu\nu|\alpha}$ as our estimate of $K_{\mu\nu|\alpha}$ at the origin of our chosen cell. Note that in this case we would solve six separate least squares problems, one for each of the six $\tilde{K}_{\mu\nu}(x^{\alpha})$. In each case we need to compute just four numbers, one $\tilde{K}_{\mu\nu}$ and three $\tilde{K}_{\mu\nu|\alpha}$. Once again we see that the least squares method is appropriate.

3. Smooth lattice dynamics

In the standard ADM 3+1 formulation of vacuum General Relativity there are four constraint equations

$$0 = R + K^2 - K_{\mu\nu} K^{\mu\nu} \tag{3.1}$$

$$0 = K_{|\mu} - K_{\mu|\nu}^{\nu} \tag{3.2}$$

and six evolution equations

$$\frac{\partial g_{\mu\nu}}{\partial t} = -2N K_{\mu\nu} + \beta_{\mu|\nu} + \beta_{\nu|\mu}$$

$$\frac{\partial K_{\mu\nu}}{\partial t} = -N_{|\mu\nu} + N (R_{\mu\nu} + K K_{\mu\nu} - 2K_{\mu\alpha} K_{\nu}^{\alpha}) + N^{\alpha} K_{\mu\nu|\alpha} + K_{\mu\alpha} N_{|\nu}^{\alpha} + K_{\nu\alpha} N_{|\mu}^{\alpha}$$

where $K = K_{\alpha}^{\alpha}$, $R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu}$ and $R = g^{\mu\nu} R_{\mu\nu}$.

Let us concentrate, for the moment, on the evolution equations. When cast as evolution equations along the timelike normals these equations take on a particularly simple form, namely,

$$\frac{dg_{\mu\nu}}{dt} = -2N K_{\mu\nu} \quad (3.3)$$

$$\frac{dK_{\mu\nu}}{dt} = -N_{|\mu\nu} + N (R_{\mu\nu} + K K_{\mu\nu} - 2K_{\mu\alpha} K_{\nu}^{\alpha}) \quad (3.4)$$

The easiest road to these equations is to simply put the shift vector to zero in the original equations. However these equations can also be easily derived by direct calculation.

The set of equations from (1.2) through to (3.4) can be viewed as a large set of coupled ordinary differential equations controlling the future evolution of the lattice. These equations will need to be solved by some numerical integration scheme to generate the lattice at successive time steps. For pure simplicity we shall demonstrate one time step using the naive Euler method, namely, that each quantity is updated according to $x \leftarrow x + \dot{x}\delta t$. Note that we do not advocate the use of this naive integration scheme in any realistic implementation (suitable schemes would include predictor-corrector, leapfrog or Runge-Kutta methods).

We will start with the lattice data on Σ_0 , namely, all of the $L_{ij}, \theta_i, \phi_i, dL_{ij}/dt, d\theta_i/dt$ and $d\phi_i/dt$. We will also assume that well defined rules are given for choosing the N, N^{μ} and γ^{μ} on each vertex of the lattice for all time. We will finish with this data fully evolved to the next hypersurface $\Sigma_{\delta t}$.

Here is our proposed algorithm.

1. For each computational cell
 - 1.1 Use equations (1.2,1.3) to compute the $x_i^{\mu}, dx_i^{\mu}/dt$ and the $R_{\mu\nu\alpha\beta}$.
 - 1.2 Use equations (1.4) and its time derivatives to compute the α_{ij} and $d\alpha_{ij}/dt$.
 - 1.3 Use equations (2.3,2.6) to compute the dL'_{ij}/dt and $d\alpha'_{ij}/dt$
 - 1.4 Use equations (2.9,2.10) to compute the $K_{\mu\nu}$
 - 1.5 Use equations (2.11,2.12) to compute $N_{|\mu\nu}$.
 - 1.6 Update L_{ij} using dL_{ij}/dt .
 - 1.7 Update x_i^{μ} using dx_i^{μ}/dt .
 - 1.8 Update θ_i, ϕ_i using $d\theta_i/dt, d\phi_i/dt$.

- 1.9 Update $K_{\mu\nu}$ using equations (3.4).
2. Interpolate $K_{\mu\nu}$ back to the primary lattice.
3. For each computational cell
 - 3.1 Update dL'_{ij}/dt using equations (2.9).
 - 3.2 Update dL_{ij}/dt using equations (2.3).
 - 3.3 Update $d\alpha'_{ij}/dt$ using equations (2.10).
 - 3.4 Update $d\alpha_{ij}/dt$ using equations (2.6).
 - 3.5 Update $d\theta_i/dt, d\phi_i/dt$ using the time derivative of equations (1.4).

There are a number of points that need to be made. As each leg will appear in one or more computational cells there is an ambiguity in how L_{ij} and dL_{ij}/dt should be updated. For example, in the cubic lattice of Fig (5), each leg is contained in exactly two different computational cells. In this case the L_{ij} and dL_{ij}/dt could be updated as the average from both cells. Other options are possible and should be explored by direct numerical experimentation. Notice that there is no such ambiguity in updating the angles α_{ij} since these are defined at the origin of each computational cell.

The calculations required in step 2 are non-trivial and arise because the shadow lattice will drift from the primary lattice (see Fig (3)). As with the computation of the $K_{\mu\nu|\alpha}$ we would need to use coordinate transformations between neighbouring cells and a least squares linear approximation to interpolate the $K_{\mu\nu}$ from the shadow lattice back to the primary lattice. If one chooses the drift vector to be zero at the origin of each cell then this step is not needed and steps 1 and 3 can be merged as one larger step. This is a considerable computational advantage. But does it restrict the class of spacetimes that can be built by this method? Setting the drift vector to zero everywhere on a lattice is much the same as setting the shift vector to zero everywhere on a finite difference grid. We know that in the later case the lapse function may need to be carefully chosen so as to avoid the numerical grid collapsing on itself (eg. as occurs for $N = 1, N^\mu = 0$ in a Schwarzschild spacetime). The same should apply to a lattice – the lapse will need to be carefully chosen to ensure that the lattice does not collapse on itself. There are also good arguments for using a shift vector to encourage the grid points to maintain a reasonable structure (eg. that the density of grid points does not drift too far from some preferred arrangement). It seems reasonable to expect that many interesting spacetimes can be explored using a zero drift vector and an appropriately chosen lapse function. Despite this, there are known cases (in finite difference calculations) where a non-zero shift vector is highly desirable. For example, in the apparent horizon boundary condition used by Anninos et al [3] a non-zero shift vector is explicitly used to ensure that grid points do not fall into a black hole and to retain a good resolution of the metric in the

vicinity of the apparent horizon. Such a condition led to a dramatic improvement in the long term stability of the numerical evolution. We can expect that a similar condition will be needed in the smooth lattice approach (though this has yet to be tested).

Another important issue is to what extent does the extensive use of least squares approximations (which are required in steps 1.1, 1.4, 1.5, 2, 3.5) effect the accuracy and stability of the evolution? The least squares approximations are bound to introduce some degree of smoothing in the estimates of $N_{|\mu\nu}, K_{\mu\nu}$ etc. Does this wash out unimportant numerical noise or important short scale variations in the lattice data? Most likely both important and unimportant information is lost. The question is does this effect the long term stability of the evolution? Again this can be investigated by way of a number of numerical experiments on a series of successively refined lattices.

The reason that we advocated the use of least squares was that it overcame the problems of estimating various quantities from an overdetermined system of equations. One could imagine a highly idealised case of an overdetermined system which admits a well defined consistent solution. For example four points in a plane constitute an overdetermined system for a straight line. But properly chosen those four points will determine one straight line. In all other cases a straight line approximation could be constructed by a least squares method. Suppose now that we wished to impose some dynamics on this toy problem. Let us suppose that we started with four points that lie on one straight line (ie. a consistent yet overdetermined system). If we impose individual evolution equations for each of the four points then those four points may no longer consistently determine a unique straight line. If however we impose the evolution equations on the parameters of the straight line then we can consistently evolve all four points. This same observation applies to our evolution of the smooth lattice. It would be wrong to search directly for evolution equations for each of the leg lengths. In our scheme we impose the dynamics on the $K_{\mu\nu}$ which we extract from an overdetermined system of equations on the lattice data. Our hope is that if we start with a lattice for which we have a good least squares fit for $K_{\mu\nu}$ etc. then the evolution will continue to yield a good fit. Again this is speculative and must be tested by direct numerical computation.

The discussion so far has dealt entirely with the evolution of the lattice. Some words are thus in order regarding the construction of initial data on the lattice. We have already seen how all of the quantities needed for the constraint equations can be extracted from the lattice. Thus we are able to evaluate the right hand side of the constraint equations. Should this be non-zero then clearly we do not have valid initial data and some corrections must be made. Following the philosophy espoused in the previous paragraph the corrections to the lattice data should be driven by corrections in the coordinate data. How this might be achieved is far from clear. Can an approach similar to York's conformal method be used? Part of

that method is to propose a 3-metric of the form $g_{\mu\nu} = \phi^4 \tilde{g}_{\mu\nu}$ where $\tilde{g}_{\mu\nu}$ is a given 3-metric (commonly chosen to be flat) and ϕ is the conformal factor. With two metrics one has a choice as to which should be expressed in Riemann normal form. If we choose \tilde{g} then we can imagine also having a conformal lattice with leg lengths \tilde{L}_{ij} . The L_{ij} of the physical lattice could then be estimated as $L_{ij} = \phi_i \phi_j \tilde{L}_{ij}$ (as suggested by Piran and Williams [4]). It is unclear at present how York's transverse traceless tensor can be represented on the conformal lattice. On the other hand if one chooses to express the physical metric $g_{\mu\nu}$ in Riemann normal form then what form does a flat $\tilde{g}_{\mu\nu}$ take in these coordinates? We conclude that it may not be a simple task to apply York's method to a smooth lattice.

Another important question concerns the order of the discretisation errors. Previous calculations (see [1]) showed that the smooth lattice method yielded $\mathcal{O}(\epsilon^2)$ accurate estimates for both the metric and the Riemann tensors. However, this is unlikely to be the case for generic lattices. The formal truncation error in the Taylor series expansion of the metric, equation (1.1), is $\mathcal{O}(\epsilon^3)$ and thus we can expect the discretisation error in the curvatures to be of $\mathcal{O}(\epsilon^1)$. This will couple to the smooth lattice equations leading to an expected error in the metric of $\mathcal{O}(\epsilon^1)$. This is less than optimal – we would prefer an $\mathcal{O}(\epsilon^2)$ error for generic lattices. The challenge then is to find ways in which second order accurate estimates for the curvatures on the lattice can be obtained. This also applies to other term such as $N_{|\mu\nu}$ and $K_{\alpha\beta|\mu}$. We hope to report on this in a later paper.

4. An example : The Kasner cosmology

The Kasner metric

$$ds^2 = -(dt')^2 + f_x^2(t')(dx')^2 + f_y^2(t')(dy')^2 + f_z^2(t')(dz')^2$$

describes a homogeneous cosmology with a T^3 topology. This metric, for the particular choice $f_x(t') = t'^{p_x}$, $f_y(t') = t'^{p_y}$, $f_z(t') = t'^{p_z}$, is a solution of the vacuum Einstein equations when

$$1 = p_x + p_y + p_z = p_x^2 + p_y^2 + p_z^2$$

A trivial transformation of the spatial coordinates

$$\begin{aligned} x &= f_x(t')(x' - x'_o) \\ y &= f_y(t')(y' - y'_o) \\ z &= f_z(t')(z' - z'_o) \end{aligned}$$

where $(x, y, z)'_o$ is any nominated point, leads to a 3-metric which, at the chosen point, is in Riemann normal form with $g_{ij} = \text{diag}(1, 1, 1)$. This fact and the clear simplicity of the

metric suggests that the 3+1 smooth lattice approach should yield accurate approximations to the Kasner metric.

In fact this problem is so simple that we will be able to show that there is no discretisation error. Of course this fortuitous result can not be expected to occur in other less symmetric spacetimes.

The 3-dimensional lattice is chosen to be a cubic lattice with the addition of two extra diagonal legs to each 2-dimensional face. The typical RNC cell will then consist of the central vertex, its six nearest neighbours, the six legs that join them and the twelve diagonal legs not attached to the central vertex (see Figs (4–5)). The data for each RNC cell will be the 6 radial leg lengths L_{oi} , 12 diagonal leg lengths L_{ij} , the 6 angles α_{ij} and their respective time derivatives, dL_{oi}/dt , dL_{ij}/dt and $d\alpha_{ij}/dt$.

Throughout the evolution we will impose the following gauge freedoms.

- ◊ The lapse function has the value one everywhere, $N_i = 1$.
- ◊ The shift vector is zero at the origin of each RNC, $0 = N_o^\mu$.
- ◊ The drift vector is zero everywhere, $0 = \gamma_i^\mu$.

While on the initial $t = t_0$ hypersurface we will assume

- ◊ The diagonal legs are constrained by $L_{ij}^2 = L_{oi}^2 + L_{oj}^2$.
- ◊ The angles (θ_i, ϕ_i) are constrained such that $\alpha_{ij} = \pi/2$ and $d\alpha_{ij}/dt = 0$.

The choice $\alpha_{ij} = \pi/2$ corresponds to placing the vertices on the coordinate axes while $d\alpha_{ij}/dt = 0$ ties them to the axes (though they are free to move subsequently off the axes). The constraint on the diagonals ensures that the 3-lattice is initially flat, ie. $R_{\mu\nu\alpha\beta} = 0$.

We will defer stating the homogeneity conditions until we have explicitly calculated all of the $K_{\mu\nu}$.

Starting with equations (1.2–2.1) we quickly obtain

$$R_{\mu\nu\alpha\beta} = 0$$

$$\begin{aligned} (x_{1+}^\mu) &= (L_{1+}, 0, 0) & (x_{1-}^\mu) &= (-L_{1-}, 0, 0) \\ (x_{2+}^\mu) &= (0, L_{2+}, 0) & (x_{2-}^\mu) &= (0, -L_{2-}, 0) \\ (x_{3+}^\mu) &= (0, 0, L_{3+}) & (x_{3-}^\mu) &= (0, 0, -L_{3-}) \\ (N_{1+}^\mu) &= (\dot{L}_{1+}, 0, 0) & (N_{1-}^\mu) &= (-\dot{L}_{1-}, 0, 0) \\ (N_{2+}^\mu) &= (0, \dot{L}_{2+}, 0) & (N_{2-}^\mu) &= (0, -\dot{L}_{2-}, 0) \\ (N_{3+}^\mu) &= (0, 0, \dot{L}_{3+}) & (N_{3-}^\mu) &= (0, 0, -\dot{L}_{3-}) \end{aligned}$$

where a dot denotes d/dt .

Since $\gamma_i^\mu = 0$ for $t \geq t_0$ while $d\alpha_{ij}/dt = 0$ on $t = t_0$ we find from (2.2–2.3) that

$$\begin{aligned}\frac{dx_i'^\mu}{dt} &= \frac{dx_i^\mu}{dt} && \text{for } t \geq t_0 \\ \frac{dL'_{ij}}{dt} &= \frac{dL_{ij}}{dt} && \text{for } t \geq t_0\end{aligned}$$

while

$$\frac{d\alpha'_{ij}}{dt} = \frac{d\alpha_{ij}}{dt} = 0 \quad \text{for } t = t_0$$

Using the above we find that equations (2.9–2.10) may be reduced to

$$L_{ij} \frac{dL_{ij}}{dt} = -K_{\mu\nu} \Delta x_{ij}^\mu \Delta x_{ij}^\nu \quad (2.9')$$

$$0 = -2K_{\mu\nu} \Delta x_{oi}^\mu x_{oj}^\nu \quad (2.10')$$

on $t = t_0$. From (2.10') we see immediately that $K_{\mu\nu} = 0$ for $\mu \neq \nu$. The remaining equations are of the form

$$\begin{aligned}L_{oi^\pm} \frac{dL_{oi^\pm}}{dt} &= -K_{ii} L_{oi^\pm}^2 \\ L_{i^\pm j^\pm} \frac{dL_{i^\pm j^\pm}}{dt} &= -K_{ii} L_{oi^\pm}^2 - K_{jj} L_{oj^\pm}^2\end{aligned}$$

for $i \neq j$ drawn from the set $\{1, 2, 3\}$. We can see that the second equation is a trivial consequence of the first equation and the assumed constraint on the diagonals. Thus this second equation is redundant and may be excluded when solving for the $K_{\mu\nu}$ on $t = t_0$. Thus we obtain the following equations for the $K_{\mu\nu}$

$$\begin{aligned}L_{01^+} \frac{dL_{01^+}}{dt} &= -K_{11} L_{01^+}^2 && L_{01^-} \frac{dL_{01^-}}{dt} = -K_{11} L_{01^-}^2 \\ L_{02^+} \frac{dL_{02^+}}{dt} &= -K_{22} L_{02^+}^2 && L_{02^-} \frac{dL_{02^-}}{dt} = -K_{22} L_{02^-}^2 \\ L_{03^+} \frac{dL_{03^+}}{dt} &= -K_{33} L_{03^+}^2 && L_{03^-} \frac{dL_{03^-}}{dt} = -K_{33} L_{03^-}^2\end{aligned}$$

This is clearly an overdetermined system for K_{11}, K_{22} and K_{33} . One could choose to take

an average solution, such as,

$$\begin{aligned} K_{11} &= \frac{-1}{2} \left(\frac{1}{L_{01+}} \frac{dL_{01+}}{dt} + \frac{1}{L_{01-}} \frac{dL_{01-}}{dt} \right) \\ K_{22} &= \frac{-1}{2} \left(\frac{1}{L_{02+}} \frac{dL_{02+}}{dt} + \frac{1}{L_{02-}} \frac{dL_{02-}}{dt} \right) \\ K_{33} &= \frac{-1}{2} \left(\frac{1}{L_{03+}} \frac{dL_{03+}}{dt} + \frac{1}{L_{03-}} \frac{dL_{03-}}{dt} \right) \end{aligned}$$

(which we would also get from a least square or singular value decomposition). However, we have yet to impose precisely the condition that the lattice be homogeneous.

Consider two neighbouring vertices p and \bar{p} and their respective RNC cells. Let x^μ and \bar{x}^μ be the RNC coordinates for the vertices p and \bar{p} respectively. We can align the coordinates such that the two vertices lie on, say, the x -axis. We can now impose homogeneity by demanding that $g_{\mu\nu}$ and $K_{\mu\nu}$ of the RNC for \bar{p} be obtained by Lie dragging the $g_{\mu\nu}$ and $K_{\mu\nu}$ of the RNC for p along the integral curves of $\partial/\partial x$. For our lattice the 3-metric is flat everywhere so this condition reduces to just a statement about the extrinsic curvatures,

$$K_{\mu\nu}(p) = \bar{K}_{\mu\nu}(\bar{p})$$

This now forces us to make an alternative choice in solving the above system for $K_{\mu\nu}$, namely,

$$K_{11} = \frac{-1}{L_{01+}} \frac{dL_{01+}}{dt} = \frac{-1}{L_{01-}} \frac{dL_{01-}}{dt} \quad (4.1)$$

$$K_{22} = \frac{-1}{L_{02+}} \frac{dL_{02+}}{dt} = \frac{-1}{L_{02-}} \frac{dL_{02-}}{dt} \quad (4.2)$$

$$K_{33} = \frac{-1}{L_{03+}} \frac{dL_{03+}}{dt} = \frac{-1}{L_{03-}} \frac{dL_{03-}}{dt} \quad (4.3)$$

The initial data must be chosen to obey these constraints.

A further consequence of this definition of homogeneity is that

$$0 = K_{\mu\nu,\alpha} = K_{\mu\nu|\alpha}$$

the second equality following from the RNC condition that $\Gamma_{\alpha\beta}^\mu = 0$ at the origin. Thus we see that the momentum constraint is identically satisfied. Since $K_{\mu\nu}$ is diagonal we also find that the Hamiltonian constraint (3.1) is just

$$0 = K_{11}K_{22} + K_{11}K_{33} + K_{22}K_{33} \quad (4.4)$$

The main evolution equations (3.4) are now

$$\frac{dK_{11}}{dt} = (-K_{11} + K_{22} + K_{33})K_{11} \quad (4.5)$$

$$\frac{dK_{22}}{dt} = (+K_{11} - K_{22} + K_{33})K_{22} \quad (4.6)$$

$$\frac{dK_{33}}{dt} = (+K_{11} + K_{22} - K_{33})K_{33} \quad (4.7)$$

while

$$\frac{dK_{\mu\nu}}{dt} = 0 \quad (4.8)$$

for $\mu \neq \nu$.

We can see that these equations, when coupled with (4.1–4.3), provide second order evolution equations for the L_{oi} . We also need second order evolution equations for α_{ij} and L_{ij} .

Using $K_{\mu\nu} = 0, dK_{\mu\nu}/dt = 0$ on $t = t_0$ for $\mu \neq \nu$ we find, by differentiation of (2.10'), that

$$\frac{d^2\alpha_{ij}}{dt^2} = 0 \quad (4.9)$$

on $t = t_0$, while differentiation of (2.9') leads to

$$\frac{d}{dt} \left(L_{ij} \frac{dL_{ij}}{dt} \right) = \frac{d}{dt} \left(L_{oi} \frac{dL_{oi}}{dt} + L_{oj} \frac{dL_{oj}}{dt} \right) \quad (4.10)$$

also on $t = t_0$. These are the required evolution equations for α_{ij} and L_{ij} . These equations are nothing more than the second time derivatives of the original constraints on the angles and diagonals, namely, $\alpha_{ij} = \pi/2$ and $L_{ij}^2 = L_{oi}^2 + L_{oj}^2$.

We are now in a position to fully analyse the future evolution of the lattice. For our lattice the initial data are $L_{oi}, L_{ij}, \alpha_{ij}$ and their first time derivatives. The complete set of evolution equations are (4.1–4.3, 4.5–4.10). There is only one initial value constraint, equation (4.4).

From equations (4.9) and (4.10) we see that our initial constraints on α_{ij} and the diagonals L_{ij} are preserved by the evolution equations. Equation (4.8) shows that $K_{\mu\nu}$ remain diagonal. The dL_{oi}/dt are updated from equations (4.1–4.3) and thus the lattice remains homogeneous.

In summary, the smooth lattice equations guarantee that the future 3-dimensional lattices remain flat and homogeneous.

By analogy with the Kasner form of the metric we might seek a solution of the form

$$\begin{aligned} L_{01+} &= L_{01-} = At^{p_x} \\ L_{02+} &= L_{02-} = Bt^{p_y} \\ L_{03+} &= L_{03-} = Ct^{p_z} \end{aligned}$$

where A, B, C are constants. It can be seen that this is a solution of the constraint (4.4) and the evolution equations (4.5–4.7) provided

$$1 = p_x + p_y + p_z = p_x^2 + p_y^2 + p_z^2$$

in exact agreement with the Kasner metric.

We conclude that for this simple model there is no discretisation error. It should be noted that the same observation would occur if one had employed a finite difference approach. This is a rather trivial example as it does not touch upon some of the more delicate aspects of the method. The extrinsic curvatures were computed without resort to least squares and the assumption of homogeneity allowed us to set $N_{|\mu\nu}$ and $K_{\mu\nu|\alpha}$ to zero. There was also no need to interpolate the future values of $K_{\mu\nu}$ back to the updated lattice. In a more general setting each of these aspects can be expected to require some attention.

5. Discussion

There have been other attempts to adapt the ADM 3+1 equations to a lattice. In particular there are the works of Piran and Williams [4] and Friedmann and Jack [5]. In both cases the equations were presented in the context of the Regge Calculus. This is another lattice method in which the metric is assumed to be flat inside each pair of adjacent tetrahedra. The principle difference between their approach and ours is in the way in which the ADM equations were imposed on the lattice. In Piran+Williams and Friedmann+Jack the equations of motion for the lattice were obtained from the ADM 3+1 action principle. They were unable to evaluate the action directly because in the Regge Calculus one does not have direct access to quantities such as $R_{\mu\nu}, N_{|\mu\nu}$ etc. Instead one has pure scalar quantities such as the defect angle, the areas and volumes of simplices. Thus in developing their equations both pairs of authors needed to make many (reasonable) assumptions about how various terms in the standard ADM action could be translated into a Regge form. This is a somewhat adhoc method and thus casts some doubt on the validity of the final equations. Indeed the equations given by Friedmann and Jack differ from those given by Piran and Williams (though they do agree for appropriate choices of lapse and shift).

In contrast our equations have been derived in a systematic manner. The use of Riemann normal coordinates has allowed us to extract in a very natural way all of the relevant coordinate data from the lattice data. This in turn has lead to a very natural adaptation of the ADM 3+1 equations to the lattice. Despite this there are still many variations that need to be explored. Some of these are of a minor numerical nature (eg. should all the dL_{ij}/dt be used in estimating $K_{\mu\nu}$ or just a limited set?). Others are potentially much more important, in particular does the use of overdetermined systems lead to errors and instabilities in the evolution of the lattice? There is also the major question of how one can solve the initial value constraint equations. Despite these concerns, we believe that the basics of our algorithm are well founded and should survive in any subsequent lattice method for numerical relativity.

However, the proof of the pudding is always in the eating. The method has been shown to have no discretisation error for the rather trivial example of the Kasner cosmology. This is encouraging but it must be noted that this example avoided many of the potential problems associated with this method. So it is imperative that the proposed method be put to a serious test. We have already successfully constructed the initial data for a time symmetric initial data slice in the Schwarzschild spacetime [1]. We are currently applying our proposed method to evolve that data. The results will be reported (soon) in a later paper.

6. References

- [1] Brewin, L.C.
Riemann normal coordinates, smooth lattices and numerical relativity.
Applied Mathematics Preprint. 97/3 also available at `gr-qc/9701057` and
`http://newton.maths.monash.edu.au:8000/preprints/slgr.ps.gz`
- [2] Brewin, L.C.
Truncation errors in estimating the curvatures of smooth lattices.
In preparation.
- [3] Anninos, P. Daues, G., Masso, J., Seidel, E. and Wai-Mo Suen.
Horizon boundary condition for black hole spacetimes.
Phys.Rev.D. Vol.51(1995) pp.5562-5578.
- [4] Piran, T. and Williams, R.M.
Three-plus-one formulation of Regge calculus.
Phys.Rev.D. Vol.33(1986) pp.1622-1633.
- [5] Friedman, J.L. and Jack, I.
3+1 Regge calculus with conserved momentum and Hamiltonian constraints.
J.Math.Phys. Vol.27(1986) pp.2973-2986.

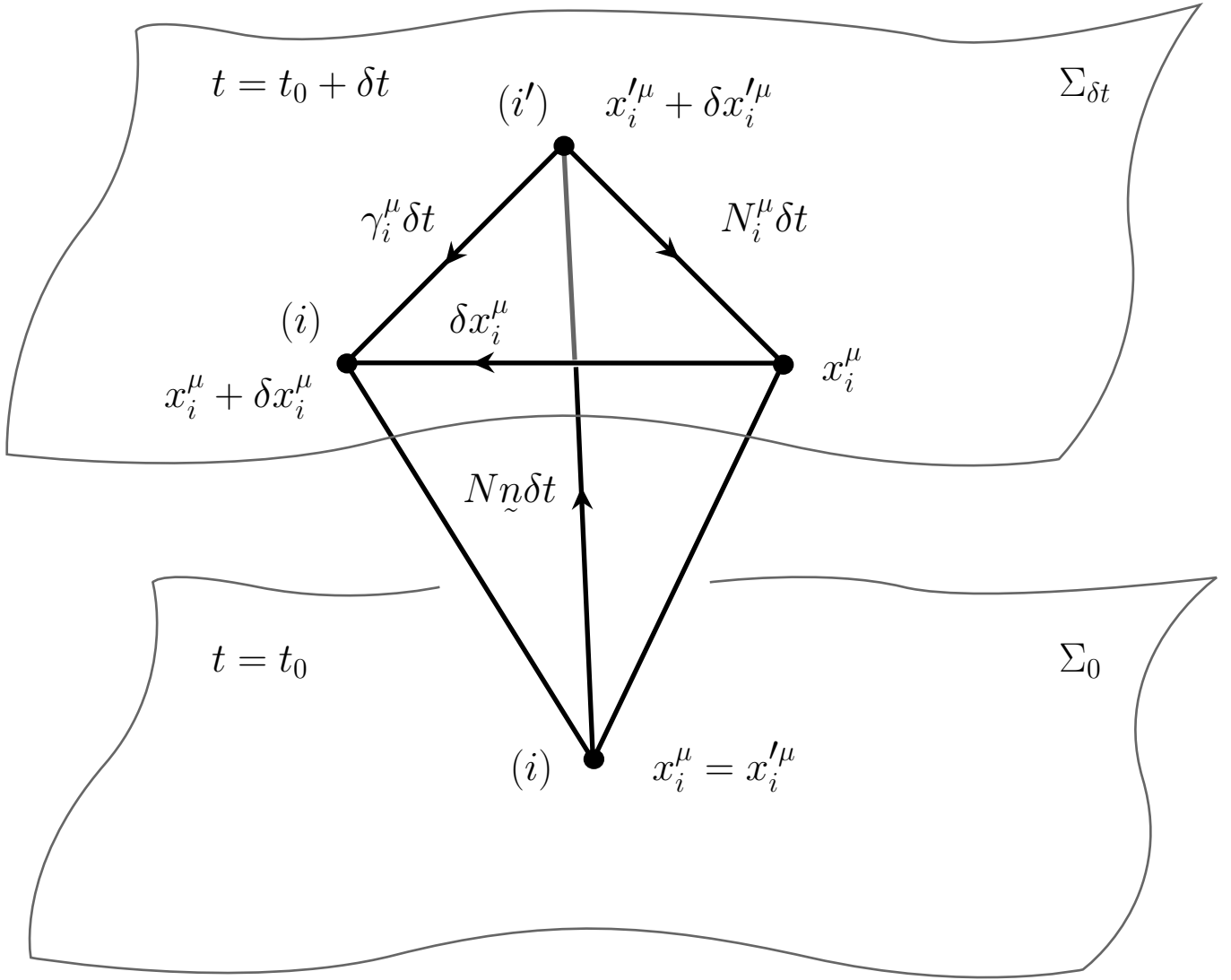


Figure 1. This figure displays the drift vector γ_i^μ and the relationships between the coordinates on the primary and shadow lattices. Clearly $\delta x_i^\mu = (-\gamma_i^\mu + N_i^\mu)\delta t$ leading directly to equation (2.1). We also have, by construction, that $x_i^\mu = x_i'^\mu$ on Σ_0 . Thus $\delta x_i^\mu = \delta x_i'^\mu + \gamma_i^\mu \delta t$ which in turn leads to equation (2.2).

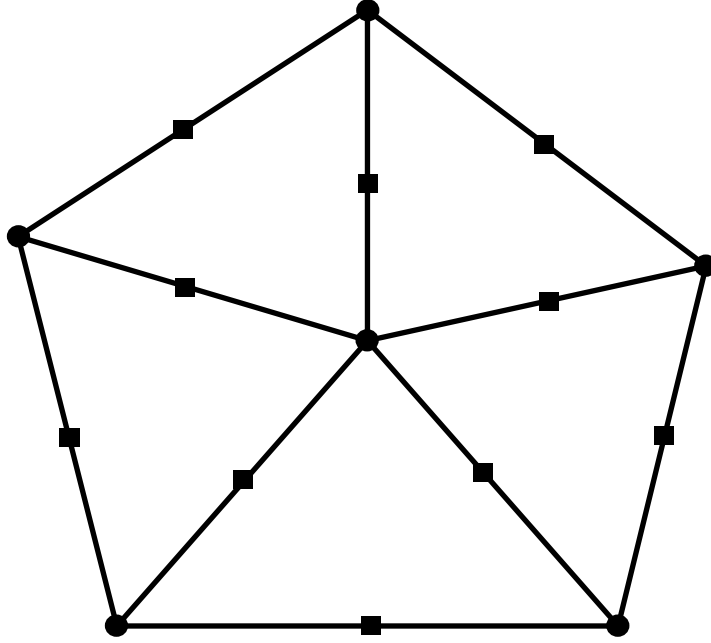


Figure 2. In the first pass over the lattice, $K_{\mu\nu}$ is computed at each vertex (\bullet). In a second pass, $K_{\mu\nu|\alpha}$ at the central vertex is estimated by fitting the linear approximation $\tilde{K}_{\mu\nu}(x) = \tilde{K}_{\mu\nu} + \tilde{K}_{\mu\nu|\alpha}x^\alpha$ to the data on each vertex. This will require a coordinate transformation from neighbouring cells to get data on the boundary of the cell. The $N_{|\mu\nu}$ at the central vertex can be estimated by first forming estimates for $N_{|\mu}$ at the centre of each leg (\blacksquare). That data is then approximated by a linear function $\tilde{N}_{|\mu}(x) = \tilde{N}_{|\mu} + \tilde{N}_{|\mu\nu}x^\nu$. We can then use $\tilde{N}_{|\mu\nu}$ as an estimate of $N_{|\mu\nu}$ at the central vertex.

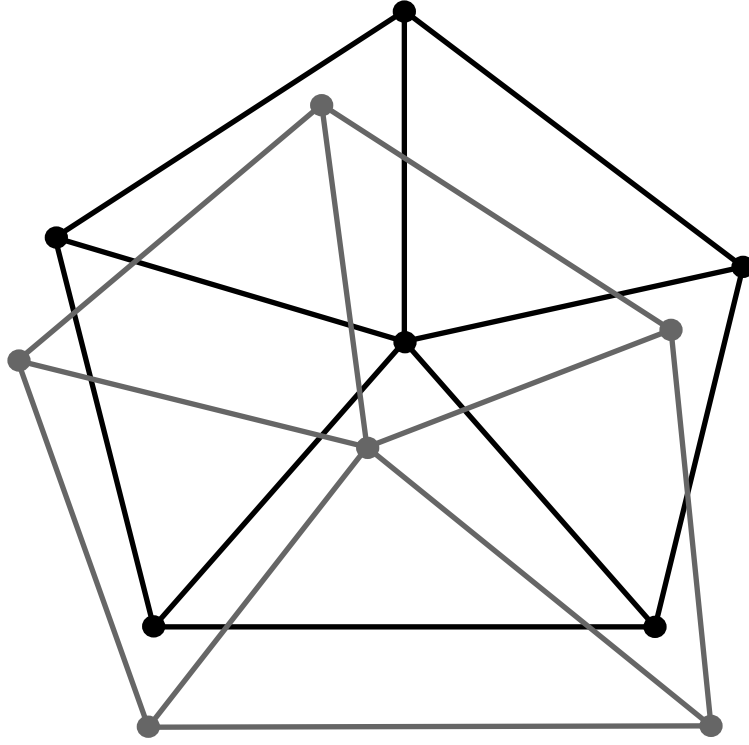


Figure 3. After one time step the shadow lattice has drifted relative to the primary lattice. However the updated $K_{\mu\nu}$ are recorded on the shadow lattice. Thus an interpolation from the shadow lattice back to the primary lattice is required. This can be done by forming a linear approximation $K_{\mu\nu}(x^\alpha) = \tilde{K}_{\mu\nu} + \tilde{K}_{\mu\nu|\alpha}x^\alpha$. and evaluating it at $x^\alpha = 0$. Note that no such interpolation is required if the drift vectors are set to zero.

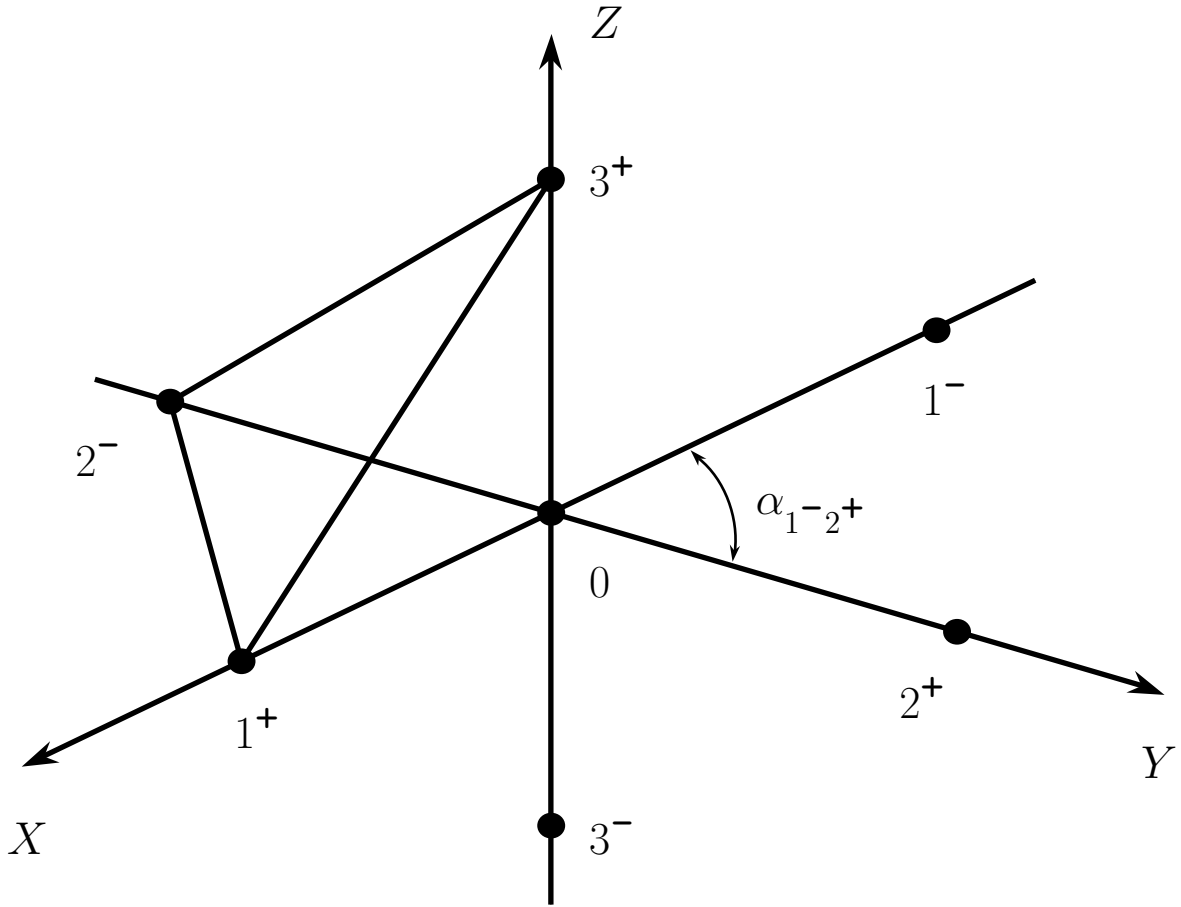


Figure 4. A typical computational cell in the cubic lattice. This cell consists of eight tetrahedra attached to the central vertex. The full cubic lattice can be obtained by replicating this structure throughout the space. Note that this leads to overlapping tetrahedra. This typical cell can be used in many spaces other than just the Kasner cosmology considered in the text. The T^3 topology of the Kasner cosmology can be obtained by identifying opposite ends of the cubic lattice. Note that, for clarity, some of the legs have not be shown in this figure (eg. (1^+3^-) , (1^-2^+) etc.)

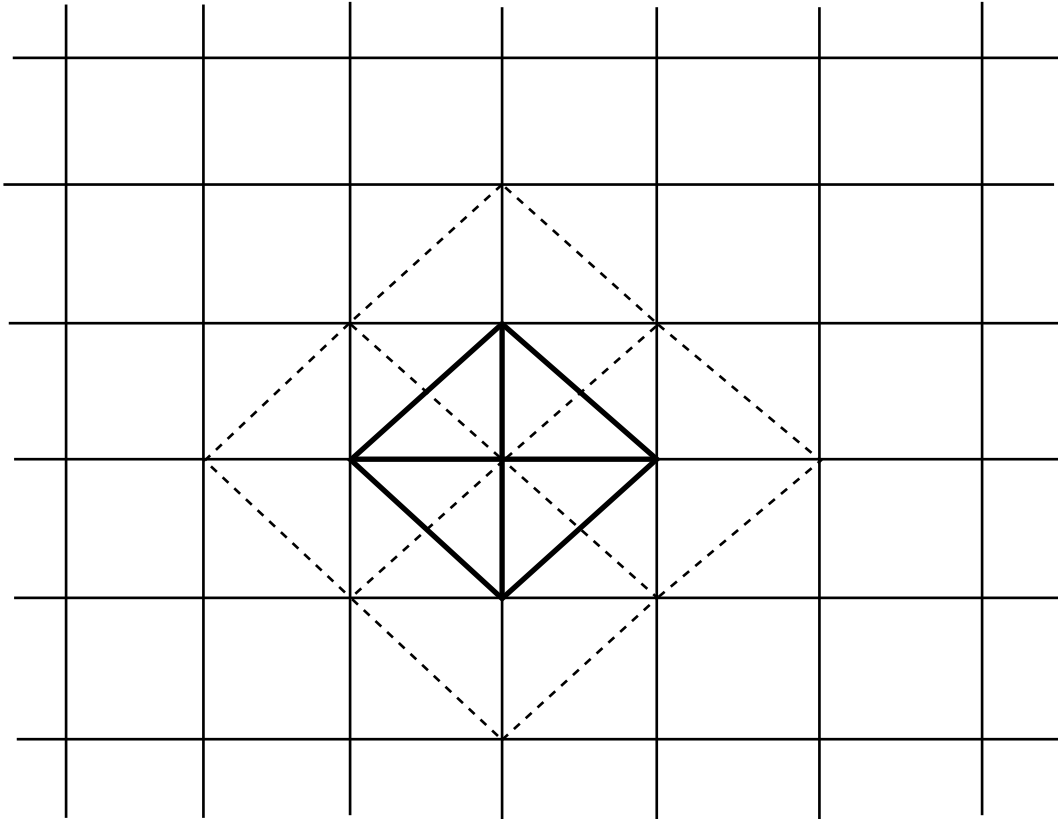


Figure 5. An xy -cross-section of (part of) the cubic lattice. The heavy solid lines denote the legs of one computational cell. Notice that the diagonal legs attached to the central vertex are not included in the computational cell. Note also how neighbouring cells overlap each other.